Math 409 midterm 2 practice #2

Name: _____

This exam has 4 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (40 pts)
In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Every subsequence of a Cauchy sequence is also a Cauchy sequence.

(b) If f is a continuous function on a closed and bounded interval I, then there exists $x_0 \in I$ such that $f(x_0)$ is the maximum of f on I.

(c) $\lim_{x \to 0} \dfrac{x^3 \sin(1/x) + x}{x \cos x}$ exists.

(d) A bounded sequence $\{x_n\}$ in $\mathbb{R}$ can have two subsequences converging to two different numbers.

(e) If $g(x) \leq -1$ for all $x \in \mathbb{R}$ and $\lim_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$.

(f) A bounded increasing sequence converges to a finite number.

(g) If $f\colon \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function, then f is bounded on $\mathbb{R}$.

(h) Given a sequence $\{x_n\}$ with $x_n > 0$ for all n, if $\{x_n\}$ has no converging subsequences, then $x_n \to \infty$, as $n \to \infty$.

Question 2. (20 pts)

(a) State the definition of Cauchy sequences.

(b) Let $\{x_n\}$ be a real sequence such that

$$x_{n+1} = x_n + \left(\frac{1}{3}\right)^n.$$

Prove that $\{x_n\}$ is a Cauchy sequence.

Question 3. (20 pts)

(a) State the Intermediate Value Theorem.

(b) Prove that there exists an $x \in \mathbb{R}$ such that $4^x = x^3 + \sin x + x^2 + 2$.

Question 4. (20 pts)

(a) State the definition of uniform continuity.

(b) Let f and g be uniformly continuous functions on $\mathbb{R}$. Prove that $f + g$ is uniformly continuous on $\mathbb{R}$.

(c) Let f and g be uniformly continuous functions on $\mathbb{R}$. If both f and g are bounded on $\mathbb{R}$, then fg is also uniformly continuous on $\mathbb{R}$.